

# Quasi-power law ensembles

GRZEGORZ WILK<sup>†</sup>

National Centre for Nuclear Research, Department of Fundamental Research,  
Hoża 69, 00-681; Warsaw, Poland

AND

ZBIGNIEW WŁODARCZYK<sup>‡</sup>

Institute of Physics, Jan Kochanowski University,  
Świętokrzyska 15; 25-406 Kielce, Poland

Quasi-power law ensembles are discussed from the perspective of nonextensive Tsallis distributions characterized by a nonextensive parameter  $q$ . A number of possible sources of such distributions are presented in more detail. It is further demonstrated that data suggest that nonextensive parameters deduced from Tsallis distributions functions  $f(p_T)$ ,  $q_1$ , and from multiplicity distributions (connected with Tsallis entropy),  $q_2$ , are not identical and that they are connected via  $q_1 + q_2 = 2$ . It is also shown that Tsallis distributions can be obtained directly from Shannon information entropy, provided some special constraints are imposed. They are connected with the type of dynamical processes under consideration (additive or multiplicative). Finally, it is shown how a Tsallis distribution can accommodate the log-oscillating behavior apparently seen in some multiparticle data.

PACS numbers: 05.90.+m, 24.10.Pa, 13.75.Ag, 24.60.Ky

## 1. Introduction

The two most characteristic ensembles is, on one side, the one resulting in exponential (Boltzmann-Gibbs - (BG)) distributions,  $f_E(X) \sim \exp(-X/T)$ , and, on the other side, the one with power distributions,  $f_P(X) \sim X^{-\gamma}$ .

---

<sup>†</sup> e-mail: wilk@fuw.edu.pl

<sup>‡</sup> e-mail: zbigniew.wlodarczyk@ujk.edu.pl

Both are encountered in the realm of the high energy multiparticle production processes investigated in hadronic and nuclear collisions. They are connected there with, respectively, nonperturbative *soft* dynamics operating at small  $X$ 's and described by exponential distributions,  $f_E(X)$ , and with perturbative *hard* dynamics, responsible for large  $X$ 's and described by power distributions,  $f_P(X)$ . These two types of dynamics are investigated separately. It is usually assumed that they operate in distinct parts of phase space of  $X$ , separated by  $X = X_0$ . However, it was found recently that the new high energy data covering the whole of phase space (cf., for example, [1, 2, 3]) are best fitted by a simple, quasi-power law formula extrapolating between both ensembles [4, 5, 6]:

$$H(X) = C \cdot \left(1 + \frac{X}{nX_0}\right)^{-n} \longrightarrow \begin{cases} \exp\left(-\frac{X}{X_0}\right) & \text{for } X \rightarrow 0, \\ X^{-n} & \text{for } X \rightarrow \infty, \end{cases} \quad (1)$$

This formula coincides with the so called Tsallis nonextensive distribution [7] for  $n = 1/(q - 1)$ ,

$$h_q(X) = C_q \left[1 - (1 - q)\frac{X}{X_0}\right]^{\frac{1}{1-q}} \stackrel{\text{def}}{=} C_q \exp_q\left(-\frac{X}{X_0}\right) \stackrel{q \rightarrow 1}{\longrightarrow} C_1 \exp\left(-\frac{X}{X_0}\right). \quad (2)$$

This is the distribution we shall concentrate on and discuss. In Section 2 we shall discuss some examples of processes leading to such distributions. It depends on the nonextensivity parameter  $q$  and this can be different depending on whether it arises from a Tsallis distribution ( $q_1$ ) or from the nonextensive Tsallis entropy ( $q_2$ ). Both are connected by  $q_1 + q_2 = 2$  and this relation seems to be confirmed experimentally. This is presented in Section 4. In Section 3 we shall discuss necessary conditions for obtaining a Tsallis distribution from Shannon information entropy. Section 5 demonstrates that a Tsallis distribution can also accommodate the log-periodic oscillations apparently observed in high energy data. Our conclusions and a summary are presented in Section 6.

## 2. Some examples of mechanisms leading to Tsallis distributions

In many practical applications, a Tsallis distribution is derived from Tsallis statistics based on his nonextensive entropy<sup>1</sup>,

$$S_q = - \sum p_i \ln_q p_i \quad \text{where} \quad \ln_q x = \frac{p^{q-1} - 1}{q - 1} \quad (3)$$

---

<sup>1</sup> Cf. [8] for most recent work with references; thermodynamical consistency of such an approach can be found in [9]. We shall not discuss this point here.

On the other hand, there are even more numerous examples of physical situations not based on  $S_q$  and still leading to quasi-power distributions in the Tsallis form. In what follows, we shall present some examples of such mechanisms, concentrating on those which allow for an interpretation of the parameter  $q$ .

### 2.1. Superstatistics

The first example is *superstatistics* [10] (cf. also [11, 12]) based on the property that a gamma-like fluctuation of the scale parameter in exponential distribution results in the  $q$ -exponential Tsallis distribution with  $q > 1$  (cf. Eq. (2)). The parameter  $q$  defines the strength of such fluctuations,  $q = 1 + \text{Var}(X) / \langle X \rangle^2$ . From the thermal perspective, it corresponds to a situation in which the heat bath is not homogeneous, but has different temperatures in different parts, which are fluctuating around some mean temperature  $T_0$ . It must be therefore described by two parameters: a mean temperature  $T_0$  and the mean strength of fluctuations, given by  $q$ . As shown in [13], this allows for further generalization to cases where one also has an energy transfer to/from heat bath. The scale  $T$  in the Tsallis distribution becomes then  $q$ -dependent:

$$T = T_{eff} = T_0 + (q - 1)T_V \quad (4)$$

Here the parameter  $T_V$  depends on the type of energy transfer, cf. [14, 15] for illustrative examples from, respectively, nuclear collisions and cosmic ray physics.

### 2.2. Preferential attachment

The second example is the *preferential attachment approach* (used in stochastic networks [16]). Here the system under consideration exhibits correlations of the preferential attachment type (like, for example, "rich-get-richer" phenomenon in networks) and the scale parameter depends on the variable under consideration. If  $x_0 \rightarrow x'_0(x) = x_0 + (q - 1)x$  then the probability distribution function,  $f(x)$ , is given by an equation the solution of which is a Tsallis distribution (again, with  $q > 1$ ):

$$\frac{df(x)}{dx} = -\frac{1}{x'_0(x)}f(x) \implies f(x) = \frac{2-q}{x_0} \left[ 1 - (1-q)\frac{x}{x_0} \right]^{\frac{1}{1-q}}. \quad (5)$$

For  $x'_0(x) = x_0$  one again gets the usual exponential distribution.

### 2.3. Multiplicative noise

Consider now a *Tsallis distribution from multiplicative noise* [11, 12]. We start from the Langevin equation [12],

$$\frac{dp}{dt} + \gamma(t)p = \xi(t) \quad (6)$$

where  $\gamma(t)$  and  $\xi(t)$  denote stochastic processes corresponding to, respectively, multiplicative and additive noises. This results in the following Fokker-Planck equation for the distribution function  $f$ ,

$$\frac{\partial f}{\partial t} = -\frac{\partial (K_1 f)}{\partial p} + \frac{\partial^2 (K_2 f)}{\partial p^2}. \quad (7)$$

Stationary  $f$  satisfies

$$\frac{d(K_2 f)}{dp} = K_1 f, \quad (8)$$

with

$$K_1 = \langle \xi \rangle - \langle \gamma \rangle p \quad \text{and} \quad K_2 = \text{Var}(\xi) - 2\text{Cov}(\xi, \gamma)p + \text{Var}(\gamma)p^2. \quad (9)$$

In the case of no correlation between noises and no drift term due to additive noise (i.e., for  $\text{Cov}(\xi, \gamma) = \langle \xi \rangle = 0$  [17]) its solution is a Tsallis distribution for  $p^2$ ,

$$f(p) = \left[ 1 + (q-1) \frac{p^2}{T} \right]^{\frac{q}{1-q}} \quad \text{with} \quad T = \frac{2\text{Var}(\xi)}{\langle \xi \rangle}; \quad q = 1 + \frac{2\text{Var}(\gamma)}{\langle \gamma \rangle}. \quad (10)$$

However, if we insist on a solution in the form of

$$f(p) = \left[ 1 + \frac{p}{nT} \right]^n, \quad n = \frac{1}{q-1}, \quad (11)$$

Eq. (8) has to be replaced by

$$K_2(p) = \frac{nT+p}{n} \left[ K_1(p) - \frac{dK_1(p)}{dp} \right]. \quad (12)$$

One then gets  $f(p)$  in the form of a Tsallis distribution, (11), but with

$$n = 2 + \frac{\langle \gamma \rangle}{\text{Var}(\gamma)} \quad \text{or} \quad q = 1 + \frac{\text{Var}(\gamma)}{\langle \gamma \rangle + 2\text{Var}(\gamma)} \quad (13)$$

and with  $q$ -dependent  $T$  (reminiscent of  $T_{eff}$  from Eq. (14) discussed before, cf. [18]):

$$T(q) = (2-q)[T_0 + (q-1)T_1] \quad \text{with} \quad T_0 = \frac{\text{Cov}(\xi, \gamma)}{\langle \gamma \rangle}, \quad T_1 = \frac{\langle \xi \rangle}{2\langle \gamma \rangle}. \quad (14)$$

#### 2.4. All variables fixed

Let us now remember that the usual situation in statistical physics is that out of three variables considered, energy  $U$ , multiplicity  $N$  and temperature  $T$ , two are fixed and one fluctuates. Fluctuations are then given by gamma distributions [19] (in the case of multiplicity distributions where  $N$  are integers, they become Poisson distributions) and only in the thermodynamic limit ( $N \rightarrow \infty$ ) does one get them in the form of Gaussian distributions, usually discussed in textbooks. In [19] we discussed in detail situations when two or all three variables fluctuate. If all are fixed we have a distribution of the type of

$$f(E) = (1 - E/U)^{N-2}. \quad (15)$$

This is nothing else but a Tsallis distribution with  $q = (N - 3)/(N - 2) < 1$ <sup>2</sup>.

#### 2.5. Conditional probability

For the *constrained systems* one gets  $q < 1$ . For example, if we have  $n$  independent energies,  $\{E_{i=1,\dots,N}\}$ , then each of them is distributed according to the Boltzman distribution,  $g_i(E_i) = (1/\lambda) \exp(-E_i/\lambda)$  (and their sum,  $E = \sum_{i=1}^N E_i$ , is distributed according to gamma distribution,  $g_N(E) = 1/[\lambda(N-1)](E/\lambda)^{N-1} \exp(-E/\lambda)$ ). However, if the available energy is limited,  $E = N\alpha = \text{const}$ , then the resulting *conditional probability* becomes a Tsallis distribution with  $q < 1$ <sup>3</sup>:

$$\begin{aligned} f(E_i|E = N\alpha) &= \frac{g_1(E_i) g_{N-1}(N\alpha - E_i)}{g_N(N\alpha)} = \frac{(N-1)}{N\alpha} \left(1 - \frac{1}{N} \frac{E_i}{\alpha}\right)^{N-2} = \\ &= \frac{2-q}{\lambda} \left[1 - (1-q) \frac{E_i}{\lambda}\right]^{\frac{1}{1-q}}, \end{aligned} \quad (16)$$

$$q = \frac{N-3}{N-2} < 1, \quad \lambda = \frac{\alpha N}{N-1}. \quad (17)$$

---

<sup>2</sup> Actually, such distributions emerges directly from calculus of probability for situation known as *induced partition* [20]. In short:  $N-1$  randomly chosen independent points  $\{U_1, \dots, U_{N-1}\}$  breaks segment  $(0, U)$  into  $N$  parts, length of which is distributed according to Eq. (15). The length of the  $k^{\text{th}}$  such part corresponds to the value of energy  $E_k = U_{k+1} - U_k$  (for ordered  $U_k$ ). One could think of some analogy in physics to the case of random breaks of string in  $N-1$  points in the energy space. Notice that induced partition differs from *successive sampling* from the uniform distribution,  $E_k \in [0, U - E_1 - E_2 - \dots - E_{k-1}]$ , which results in  $f(E) = 1/E$  [21].

<sup>3</sup> One could get a Tsallis-like distribution with  $q > 1$  only if the scale parameter  $\lambda$  would fluctuate in the same way as in the case of superstatistics, see [22].

### 2.6. Statistical physics

We end this part by a reminder of how Tsallis distribution with  $q < 1$  arises from *statistical physics considerations*. Consider an isolated system with energy  $U = \text{const}$  and with  $\nu$  degrees of freedom ( $n$  particles). Choose single degree of freedom with energy  $E$  (i.e., the remaining, or reservoir, energy is  $E_r = U - E$ ). If this degree of freedom is in a single, well defined, state then the number of states of the whole system is  $\Omega(U - E)$  and probability that the energy of the chosen degree of freedom is  $E$  is  $P(E) \propto \Omega(U - E)$ . Expanding (slowly varying)  $\ln \Omega(E)$  around  $U$ ,

$$\ln \Omega(U - E) = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{\partial^{(k)} \ln \Omega}{\partial E_r^{(k)}}, \quad \text{with} \quad \beta = \frac{1}{k_B T} \stackrel{\text{def}}{=} \frac{\partial \ln \Omega(E_r)}{\partial E_r}, \quad (18)$$

and (because  $E \ll U$ ) keeping only the two first terms one gets

$$\ln P(E) \propto \ln \Omega(E) \propto -\beta E, \quad \text{or} \quad P(E) \propto \exp(-\beta E), \quad (19)$$

i.e., a Boltzmann distribution (or  $q = 1$ ). On the other hand, because one usually expects that  $\Omega(E_r) \propto (E_r/\nu)^{\alpha_1 \nu - \alpha_2}$  (where  $\alpha_{1,2}$  are of the order of unity and we put  $\alpha_1 = 1$  and, to account for diminishing the number of states in the reservoir by one,  $\alpha_2 = 2$ ) [23], one can write

$$\frac{\partial^k \beta}{\partial E_r^k} \propto (-1)^k k! \frac{\nu - 2}{E_r^{k+1}} = (-1)^k k! \frac{\beta^{k-1}}{(\nu - 2)^k} \quad (20)$$

and write the full series for probability of choosing energy  $E$ :

$$\begin{aligned} P(E) &\propto \frac{\Omega(U - E)}{\Omega(U)} = \exp \left[ \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} \frac{1}{(\nu - 2)^k} (-\beta E)^{k+1} \right] = \\ &= C \left( 1 - \frac{1}{\nu - 2} \beta E \right)^{(\nu - 2)} = \\ &= \beta (2 - q) [1 - (1 - q) \beta E]^{\frac{1}{1-q}}; \end{aligned} \quad (21)$$

(where we have used the equality  $\ln(1+x) = \sum_{k=0}^{\infty} (-1)^k [x^{k+1}/(k+1)]$ ). This result, with  $q = 1 - 1/(\nu - 2) \leq 1$ , coincides with results from conditional probability and the induced partition.

### 2.7. Fluctuations of multiplicity $N$

Constant values of  $U$ ,  $N$  and  $T$  result in Eq. (15) and  $q < 1$ . To get larger values of  $q$ , one has to allow for fluctuations of one of the variables:  $U$ ,  $N$  or  $T$ . In superstatistics [10, 11, 12] it was  $T$  that was fluctuating, let

us now consider the example of fluctuating  $N$ . It means that, whereas for fixed  $N$  (to simplify notation we changed  $N - 2$  to  $N$  here)

$$f_N(E) = \left(1 - \frac{E}{U}\right)^N \quad \text{and} \quad U = \sum E = \text{const}, \quad (22)$$

for  $N$  fluctuating according to some  $P(N)$  the resulting distribution is

$$f(E) = \sum f_N(E)P(N). \quad (23)$$

The most characteristic for our purposes are situations provided by the, respectively, *Binomial Distribution* (BD), Eq.(24), *Poissonian Distribution* (PD), Eq.(25) and by the *Negative Binomial Distributions* (NBD), Eq.(26) (cf. [24]):

$$P_{BD}(N) = \frac{M!}{N!(M-N)!} \left(\frac{\langle N \rangle}{M}\right)^N \left(1 - \frac{\langle N \rangle}{M}\right)^{M-N}; \quad (24)$$

$$P_{PD}(N) = \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle} \quad (25)$$

$$P_{NBD}(N) = \frac{\Gamma(N+k)}{\Gamma(N+1)\Gamma(k)} \left(\frac{\langle N \rangle}{k}\right)^N \left(1 + \frac{\langle N \rangle}{k}\right)^{-k-N}. \quad (26)$$

They lead, respectively, to Tsallis distribution with  $q = 1 - 1/M < 1$ , Eq.(27), to exponential Boltzmann distribution, Eq.(28) with  $q = 1$ , and to Tsallis distribution with  $q = 1 + 1/k > 1$ , Eq.(29) ( $\beta = \langle N \rangle/U$ ):

$$f_{BD}(E) = \left(1 - \frac{\beta E}{M}\right)^M, \quad (27)$$

$$f_{PD}(E) = \exp(-\beta E), \quad (28)$$

$$f_{NBD}(E) = \left(1 + \frac{\beta E}{k}\right)^{-k}. \quad (29)$$

Note that in all three cases

$$q - 1 = \frac{\text{Var}(N)}{\langle N \rangle^2} - \frac{1}{\langle N \rangle}. \quad (30)$$

It is natural that for BD where  $\text{Var}(N)/\langle N \rangle < 1$  one has  $q < 1$ , for PD with  $\text{Var}(N)/\langle N \rangle = 1$  also  $q = 1$  and for NBD where  $\text{Var}(N)/\langle N \rangle > 1$  one has  $q > 1$  (cf., also case  $U = \text{const}$  and  $T = \text{const}$  considered in [19]).

Fluctuations of  $N$  can be translated into fluctuations of  $T$ . Notice first that [13] (cf. also [25]) NBD,  $P(N) = \Gamma(N+k)/[\Gamma(N+1)\Gamma(k)] \cdot \gamma^k (1+\gamma)^{-k-N}$ , arises also if in the Poisson multiplicity distribution,  $P(N) =$

$\bar{N}^N e^{-\bar{N}}/N!$ , one fluctuates the mean multiplicity  $\bar{N}$  using gamma distribution  $f(\bar{N}) = \gamma^k \bar{N}^{k-1}/\Gamma(k) \cdot e^{-\gamma \bar{N}}$  with  $\gamma = k/\langle \bar{N} \rangle$ <sup>4</sup>. Now, identifying fluctuations of mean  $\bar{N}$  with fluctuations of  $T$ , one can express the above observation via fluctuations of temperature. Noticing that  $\bar{\beta} = \bar{N}/U$  (i.e., that  $\langle \bar{N} \rangle = U \langle \bar{\beta} \rangle$  and  $\gamma = k/[U \langle \bar{\beta} \rangle]$ ) one can rewrite  $f(\bar{N})$  as

$$\begin{aligned} f(\bar{\beta}) &= \frac{k}{\langle \bar{\beta} \rangle \Gamma(k)} \left( \frac{k \bar{\beta}}{\langle \bar{\beta} \rangle} \right)^{k-1} \exp \left( -\frac{k \bar{\beta}}{\langle \bar{\beta} \rangle} \right) = \\ &= \frac{\left( \frac{1}{q-1} \frac{\bar{\beta}}{\langle \bar{\beta} \rangle} \right)^{\frac{1}{q-1}-1}}{(q-1) \langle \bar{\beta} \rangle \Gamma\left(\frac{1}{q-1}\right)} \exp \left( -\frac{1}{q-1} \frac{\bar{\beta}}{\langle \bar{\beta} \rangle} \right). \end{aligned} \quad (31)$$

And this is just a gamma distribution describing fluctuations of  $\beta = 1/T$  discussed in [11].

There is more to the physical meaning of  $q$ . Because  $U = \langle N \rangle T$  the heat capacity  $C$  can be written as  $1/C = dU/dT = \langle N \rangle$ . However, because in our case  $U = \text{const}$ , i.e.,  $\text{Var}(N)/\langle N \rangle^2 = \text{Var}(\beta)/\langle \beta \rangle^2$  (or  $\langle N \rangle \sim \langle \beta \rangle$  and  $\text{Var}(N) \sim \text{Var}(\beta)$ ), the formula  $q-1 = \text{Var}(N)/\langle N \rangle^2 - 1/\langle N \rangle$  obtained before coincides with a similar formula obtained in [26]:

$$q-1 = \frac{\text{Var}(\beta)}{\langle \beta \rangle^2} - \frac{1}{C} \simeq \frac{\text{Var}(T)}{\langle T \rangle^2} - \frac{1}{C}. \quad (32)$$

This can be confronted with  $q-1 = \text{Var}(\bar{N})/\langle \bar{N} \rangle^2$  from [13], which, because of the form of  $f(\bar{\beta})$  in Eq. (31), can be rewritten approximately as

$$q-1 = \frac{\text{Var}(\bar{\beta})}{\langle \bar{\beta} \rangle^2} \simeq \frac{\text{Var}(\bar{T})}{\langle \bar{T} \rangle^2}. \quad (33)$$

### 3. Tsallis distribution from Shannon entropy

Tsallis distribution is usually derived either from Tsallis entropy (via the MaxEnt variational approach, not discussed here) or from some dynamical considerations, some examples of which are presented in this paper. However, it turns out that it also emerges in a quite natural (in the same MaxEnt approach) way from Shannon entropy, provided one imposes the right constraints. In fact, as shown in [28], one can establish a transformation between these two variational problems and shown that they contain the

---

<sup>4</sup> We have two type of averages here:  $\bar{X}$  means average value in a given event whereas  $\langle X \rangle$  denotes averages over events (or ensembles).



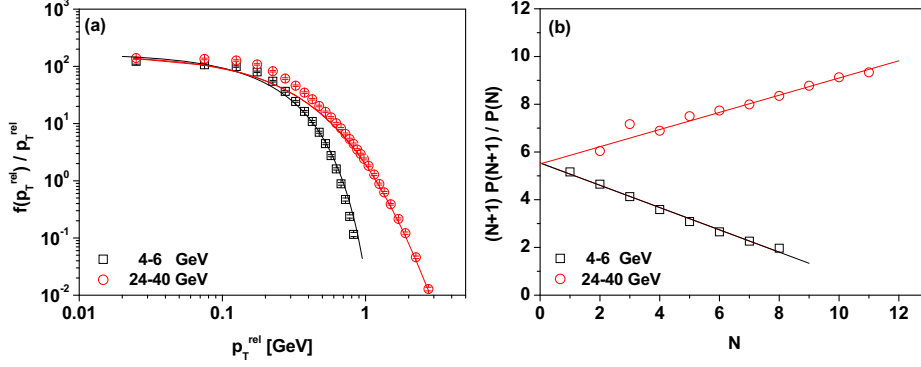


Fig. 1. (Color online) Examples of  $q < 1$  or  $q > 1$ . (a) Distributions of  $p_T^{rel} = |\vec{p} \times \vec{p}_{jet}| / |\vec{p}_{jet}|$  for particles inside the jets with different values of  $\vec{p}_{jet}$ . Distributions are fitted using a Tsallis distribution (2) with  $T = 0.18$ . (b)  $(N+1)P(N+1)/P(N) = a + bN$  as function of multiplicity  $N$  in jets with different values of  $\vec{p}_{jet}$  as presented in (a). Depending on the range of phase space covered, the corresponding values of  $q$  are 0.885 for  $\vec{p}_{jet} \in (4-6)$  GeV and 1.094 for  $\vec{p}_{jet} \in (24-40)$  GeV. Data are from [3, 27] and from the Durham HepData Project; <http://hepdata.cedar.ac.uk/view/irn9136932>.

same information. This means that the two approaches seem to be equivalent, one can either use Tsallis entropy with relatively simple constraints, or the Shannon entropy with rather complicated ones (cf., for example, a list of possible distributions one can get in this way [29]).

In general, Shannon entropy for some probability density  $f(x)$ ,  $S = -\int dx f(x) \ln[f(x)]$ , supplied with constraint  $\langle h(x) \rangle = \int dx f(x) h(x) = const$ , where  $h(x)$  is some function of  $x$ , subjected to the usual MaxEnt variational procedure, results in the following form of  $f(x)$ :

$$f(x) = \exp[\lambda_0 + \lambda h(x)], \quad (34)$$

with constants  $\lambda_0$  and  $\lambda$  calculated from the normalization of  $f(x)$  and from the constraint equation. It is now straightforward to check that

$$\langle z \rangle = z_0 = \frac{q-1}{2-q} \quad \text{where} \quad z = \ln \left[ 1 - (1-q) \frac{E}{T_0} \right] \quad (35)$$

results in  $f(z) = (1/z_0) \exp(-z/z_0)$  which translates to (remembering that  $f(z)dz = f(E)dE$ ) a Tsallis distribution

$$f(E) = \frac{1}{(1+z_0)T_0} \left( 1 + \frac{z_0}{1+z_0} \frac{E}{T_0} \right)^{-\frac{1+z_0}{z_0}} = \frac{2-q}{T_0} \left[ 1 - (1-q) \frac{E}{T_0} \right]^{\frac{1}{1-q}}. \quad (36)$$

The parameter  $T_0$  can be deduced from the additional condition which must be imposed, namely from the assumed knowledge of the  $\langle E \rangle$  (notice that in the case of BG distribution this would be the only condition).

So far the physical significance of the constraint (35) is not fully understood. Its form can be deduced from the idea of varying scale parameter in the form of the preferential attachment, Eq. (5), which in present notation means  $T \rightarrow T(E) = T_0 + (q - 1)E$ . As shown in (5) it results in Tsallis distribution (36). This suggest the use of  $z = \ln [T(E)/T_0]$  constrained as in Eq. (35). In such approach  $\ln f(E) = -[1/(q - 1)] \ln [T(E)] + [(2 - q)/(q - 1)] \ln (T_0)$  and, because  $S = -\langle \ln f(E) \rangle$ , therefore  $S = 1/(2 - q) + \ln (T_0)$  for Tsallis distribution becoming  $S = 1 + \ln (T_0)$  for Boltzmann-Gibbs (BG) distribution ( $q = 1$ ).

It is interesting that the constraint (35) seems to be natural a for multiplicative noise described by the Langevine equation:  $dp/dt + \gamma(t)p = \xi(t)$ , with traditional multiplicative noise  $\gamma(t)$  and additive noise (stochastic processes)  $\xi(t)$  (see [18] for details). In fact, there is a connection between the kind of noise in this process and the condition imposed in the MaxEnt approach. For processes described by an additive noise,  $dx/dt = \xi(t)$ , the natural condition is that imposed on the arithmetic mean,  $\langle x \rangle = c + \langle \xi \rangle t$ , and it results in the exponential distributions. For the multiplicative noise,  $dx/dt = x\gamma(t)$ , the natural condition is that imposed on the geometric mean,  $\langle \ln x \rangle = c + \langle \gamma \rangle t$ , which results in a power law distribution [30].

#### 4. Tsallis entropy vs Tsallis distributions ( $q_1 + q_2 = 2$ )

One has to start with some explanatory remarks. The Tsallis distribution can be also obtained via MaxEnt procedure from Tsallis entropy,

$$S_q = -\frac{1}{1-q} \sum (1 - p_i^q) = -\langle \ln_q p_i \rangle_q = -\langle \ln_{2-q} p_i \rangle_{q=1}, \quad (37)$$

where  $\ln_q x = (x^{1-q} - 1)/(1 - q)$  and  $\langle x \rangle_q = \sum p_i^q x_i$ . Now, depending on the condition imposed one gets from  $S_q$

$$\text{either } f(x) = q[1 + (1 - q)x]^{-\frac{1}{1-q}} \quad \text{for } \langle x \rangle_1 \quad (38)$$

$$\text{or } f(x) = (2 - q)[1 + (q - 1)x]^{\frac{1}{1-q}} \quad \text{for } \langle x \rangle_q. \quad (39)$$

However, after replacement of  $q$  by  $q_1 = 2 - q$ , the distribution (38) becomes the usual Tsallis distribution (39). Therefore one encounters an apparent puzzle, namely the  $q_1$  of Tsallis distribution does not coincides with the  $q_2$  of corresponding Tsallis entropy, instead they are connected by relation  $q_1 + q_2 = 2$ . The natural question therefore arises: is such a relation seen in data? As shown in [31] that seems really be the case, at least quantitatively.

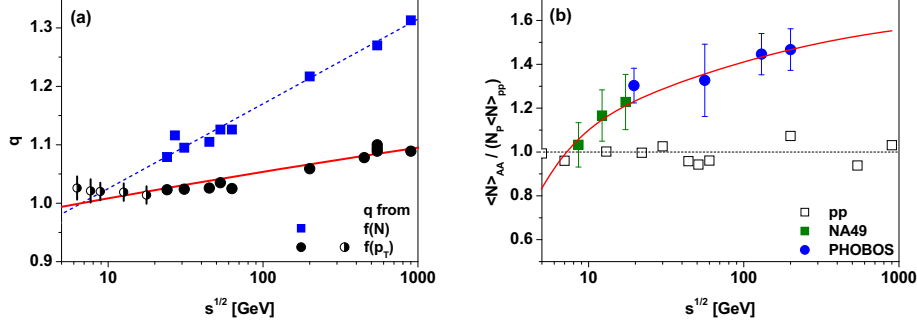


Fig. 2. (Color online) (a) Energy dependencies of the parameters  $q$  obtained from, respectively: multiplicity distributions  $P(N)$  [33] (squares), from different analysis of transverse momenta distributions  $f(p_T)$  in  $p+p$  data ([34] - circles, full symbols) and from data on  $f(p_T)$  from Pb+Pb collisions ([35] - half filled circles). (b) Energy dependence of the charged multiplicity for nucleus-nucleus collisions divided by the superposition of multiplicities from proton-proton collisions fitted to data on multiplicity taken from [35] (NA49) and from compilation [36].

This is seen when comparing  $q = q_1$  obtained from data on  $p_T$  distributions (cf. Fig. 2(a)) to  $q = q_2$  obtained from data on multiplicities in  $p-A$  collisions assuming that entropy is proportional to the number of particles produced (cf. Fig. 2(b)). Whereas  $q_1$  is deduced from a Tsallis distribution taken in one of the forms discussed above,  $q_2$  is deduced directly from the corresponding entropy  $S_q$  of the  $p-A$  collision. Assume that such collision can be adequately described by a superposition model in which the main ingredients are  $\nu$  nucleons which have interacted at least once [32]. Assume further that they are identical and independent and produce  $n_i$  secondaries of each other. As a result a  $N = \sum_{i=1}^{\nu} n_i$  are produced in one collision and the mean multiplicity is  $\langle N \rangle = \langle \nu \rangle \langle n_i \rangle$  (where  $\langle \nu \rangle$  is the mean number of nucleons participating in the collision and  $\langle n_i \rangle$  the mean multiplicity in  $i^{th}$  elementary collision. The corresponding entropy  $S_q^{(\nu)}$  of such process will then be  $q$ -sum of  $\nu$  entropies  $S_q^{(1)}$  of individual collisions and is given by:

$$S_q^{(\nu)} = \sum_{k=1}^{\nu} \frac{\nu!}{(\nu-k)!k!} (1-q)^{k-1} [S_q^{(1)}]^k = \frac{[1 + (1-q)S_q^{(1)}]^{\nu} - 1}{1-q}. \quad (40)$$

Notice that  $\ln [1 + (1-q)S_q^{(\nu)}] = \nu \ln [1 + (1-q)S_q^{(1)}]$  and  $S_q^{(\nu)} \xrightarrow{q \rightarrow 1} \nu \cdot S_1^{(1)}$ . For  $q < 1$  entropy  $S_q^{(\nu)}$  is nonextensive because  $S_q^{(\nu)} / \nu \xrightarrow{\nu \rightarrow \infty} \infty$ . For  $q > 1$

one has  $S_q^{(\nu)} \geq 0$  only for  $q < 1 + 1/S_q^{(1)}$  and  $S_q^{(\nu)}/\nu \xrightarrow{\nu \rightarrow \infty} 0$ , i.e., entropy is extensive,  $0 \leq S_q^{(\nu)}/\nu \leq S_q^{(1)}$ .

Because  $\langle N_{AA} \rangle > N_P \langle N_{pp} \rangle = \nu \langle N_{pp} \rangle$ , the nonextensivity parameter obtained from the corresponding entropies must be smaller than unity,  $q_2 < 1$ . On the other hand, all estimates of nonextensivity parameter from Tsallis distributions lead to  $q_1 > 1$ .

### 5. Dressed Tsallis distributions (log-periodic oscillations)

The pure power-like distributions are known to be in many cases decorated by specific log-periodic oscillations (i.e., multiplied by some dressing factor  $R$ ) [37]. They suggest some hierarchical fine-structure existing in the system under consideration and are usually regarded as possibly indicating some kind of multifractality in the system. Closer inspection of recent data from LHC [1, 3, 2, 38, 39] reveals that, for large transverse momenta  $p_T$ , one observes a similar effect, cf. Fig. 3. So far, the prevailing opinion is that this is just an apparatus induced artifact with no meaning. However, its persistence in the type of experiment considered, energy or type of collision process (provided that the range of  $p_T$  covered is large enough) calls for some explanation. Such an explanation was considered in [40, 41]. In [37], the only possibility investigated was to attribute such oscillations to complex values of the power index  $n$  in Eq. (1). As shown in [40] and also below, it also works in the case of quasi-power like Tsallis distributions. However, because one now also has a scale parameter  $T$  and a constant term, one can also offer another explanation: real  $n$  but log-periodically oscillating scale parameter  $T$ . This was discussed in detail in [41]. We shall present both possibilities here.

#### 5.1. Complex nonextensivity parameter $q$

For simple power laws, one has some function  $O(x)$  which is scale invariant,  $O(\lambda x) = \mu O(x)$  and  $O(x) = Cx^{-m}$  with  $m = -\ln \mu / \ln \lambda$ . However, this can be written as  $\mu \lambda^m = 1 = e^{i2\pi k}$ , where  $k$  is an arbitrary integer. We then have not a single power  $m$  but rather a whole family of complex powers,  $m_k$ , with  $m_k = -\ln \mu / \ln \lambda + i2\pi k / \ln \lambda$ . Their imaginary part signals a hierarchy of scales leading to log-periodic oscillations. This means that, in fact,  $O(x) = \sum_{k=0} w_k \text{Re}(x^{-m_k}) = x^{-\text{Re}(m_k)} \sum_{k=0} w_k \cos[\text{Im}(m_k) \ln(x)]$  (where  $w_k$  are coefficients of the expansion). This is the origin of the usual dressing factor appearing in [37] and used to describe data:

$$R(E) = a + b \cos[c \ln(E + d) + f] \quad (41)$$

(only  $w_1$  and  $w_2$  terms are kept).

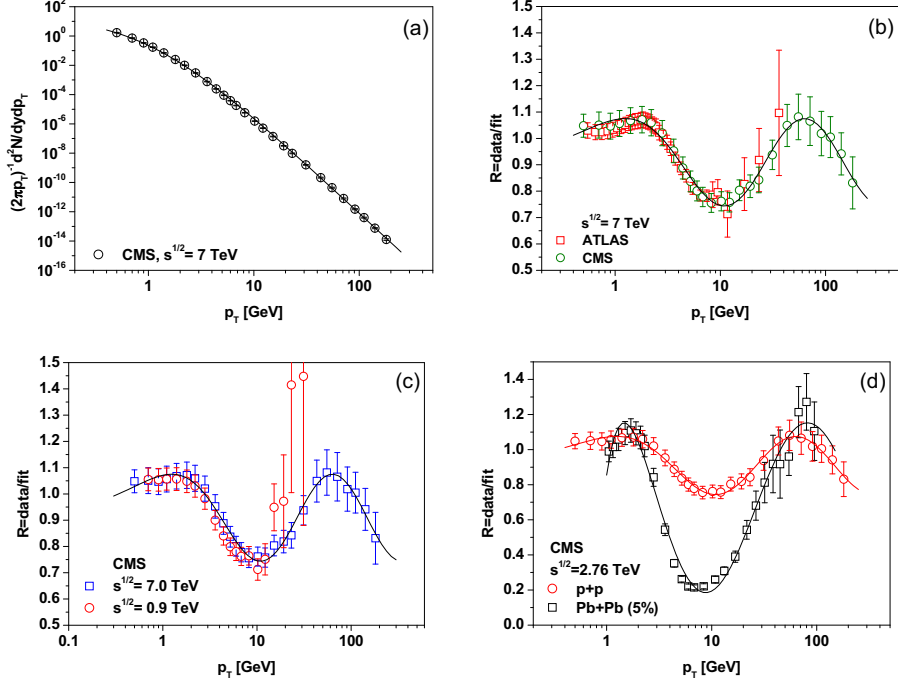


Fig. 3. (Color online) Examples of log-periodic oscillations. (a)  $dN/dp_T$  for the highest energy 7 TeV, the Tsallis behavior is evident. Only CMS data are shown [1], others behave essentially in the identical manner. (b) Log-periodic oscillations showing up in different experimental data like [1] or ATLAS [3] taken at 7 TeV. (c) Results from CMS [1] for different energies. (d) Results for different systems ( $p+p$  collisions compared with  $Pb+Pb$  taken for 5 % centrality [38]. Results from ALICE are very similar [39].

It turns out that a similar scaling solution can also be obtained in case of a Tsallis quasi-power like distribution. To this end one must start from stochastic network approach, Section 2.2 and Eq. (5), in which Tsallis distribution is obtained by introducing a scale parameter depending on the variable considered. In our case it is  $df(E)/dE = -f(E)/T(E)$  resulting in

$$f(E) = \frac{n-1}{nT_0} \left(1 + \frac{E}{nT_0}\right)^{-n} \quad \text{for} \quad T(E) = T_0 + \frac{E}{n}. \quad (42)$$

In final difference form (with change in notation:  $T_0$  replaced by  $T$ )

$$\frac{df(E)}{dE} = -\frac{f(E)}{T(E)} \implies f(E + \delta E) = \frac{-n\delta E + nT + E}{nT + E} f(E). \quad (43)$$

We consider a situation in which  $\delta E = \alpha nT(E) = \alpha(nT + E)$ . It depends now on the new scale parameter  $\alpha$  ( $\alpha < 1/n$  in order to keep changes of  $\delta E$  to be of the order of  $T$ ) and can be very small but always remains finite. It can now be shown that  $f[E + \alpha(nT + E)] = (1 - \alpha n)f(E)$  which, when expressed in the new variable  $x = 1 + E/(nT)$ , corresponds formally to the following scale invariant relation:

$$g[(1 + \alpha)x] = (1 - \alpha n)g(x). \quad (44)$$

Following the same procedure used to obtain dressed solutions of scale invariant functions discussed at the beginning of this Section, one arrives at the dressed Tsallis distribution (we keep, as before, only the two lowest terms,  $k = 0$  and  $k = 1$ )<sup>5</sup>:

$$g(E) \simeq \left(1 + \frac{E}{nT}\right)^{-m_0} \left\{ w_0 + w_1 \cos \left[ \frac{2\pi}{\ln(1 + \alpha)} \ln \left(1 + \frac{E}{nT}\right) \right] \right\}. \quad (45)$$

with  $m_0 = -\ln(1 - \alpha n)/\ln(1 + \alpha) \xrightarrow{\alpha \rightarrow 0} n$ . In addition to the scale parameter  $\alpha$  one has two more parameters occurring in the dressing factor  $R$ ,  $w_0$  and  $w_1$ . The other parameters occurring in Eq. (41) are expressed by the original parameters in the following way:  $a/b = w_0/w_1$ ,  $c = 2\pi/\ln(1 + \alpha)$ ,  $d = nT$  and  $f = -2\pi \ln(nT)/\ln(1 + \alpha) = -c \ln d$ . One can, however, consider a more involved evolution process, with  $\kappa$  sequential cascades; in this case the additional parameter  $\kappa$  changes parameter  $c$  in (41),  $c \rightarrow c' = c/\kappa$ . It does not affect the slope parameter  $m_0$  but changes the frequency of oscillations which now decrease as  $1/\kappa$ . Comparison with data requires  $\kappa \sim 22$  (cf. [40] for details).

### 5.2. Log-periodically oscillating $T$

As mentioned before, one can translate a dressed Tsallis distribution into a normal one but with a log-periodically oscillating in  $p_T$  scale factor  $T$ , cf. Fig. 4a. The formula used there to fit the obtained results resembles that for dressing factor (41),

$$T = \bar{a} + \bar{b} \sin [\bar{c} (\ln(E + \bar{d}) + \bar{f})]. \quad (46)$$

---

<sup>5</sup> Notice that in Eq. (45)  $n \neq m_0$ . However,  $n$  and  $T$  are both unknown *a priori* parameters. Therefore, for fitting purposes, where we have to use two (and not three) parameter Tsallis distribution, we use  $[1 + E/(m_0 T')]^{-m_0}$  with fitting parameters  $m_0$  and  $T'$ . In terms of  $T$ ,  $n$  and  $\alpha$  we have  $T' = nT/m_0 = -nT \ln(1 + \alpha)/\ln(1 - \alpha n)$ .

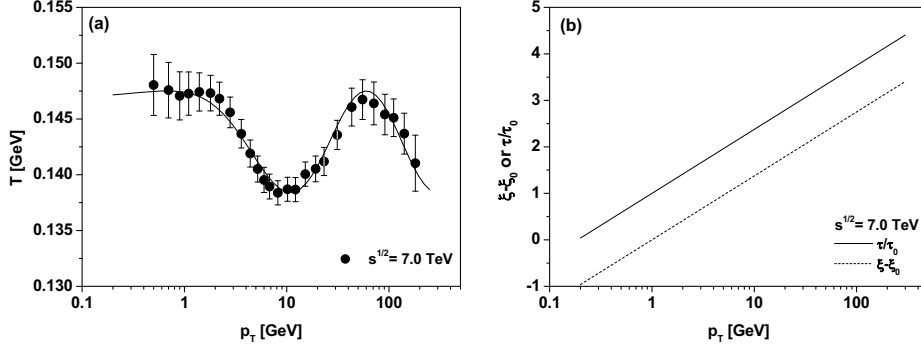


Fig. 4. (Color online) (a) Oscillations of scale parameter  $T$  leading to identical dressed Tsallis distribution as shown in Fig. 3b (obtained for CMS data at 7 TeV and fitted using Eq. (46)). (b) Dependencies of  $\tau/\tau_0$  from Eq. (54) and  $\xi - \xi_0$  from Eq. (51) resulting in oscillations of  $T$  shown in panel (a).

In fit shown in Fig. 4a parameters (generally energy dependent) are  $\bar{a} = 0.143$ ,  $\bar{b} = 0.0045$ ,  $\bar{c} = 2.0$ ,  $\bar{d} = 2.0$ ,  $\bar{f} = -0.4$ .

To explain Eq. (46) one uses a stochastic equation for the temperature evolution [42] written in Langevin formulation with energy dependent noise,  $\xi(t, E)$ , and allowing for time dependent  $E = E(t)$ <sup>6</sup>:

$$\frac{dT}{dE} \frac{dE}{dt} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (47)$$

Assuming now a scenario of *preferential attachment* (cf. Section 2.2 above) known from the growth of networks [16]) one has

$$\frac{dE}{dt} = \frac{E}{n} + T. \quad (48)$$

and Eq. (47) has now the form:

$$\left( \frac{E}{n} + T \right) \frac{dT}{dE} + \frac{1}{\tau} T + \xi(t, E) T = \Phi. \quad (49)$$

After straightforward manipulations (cf. [40] for details) one gets, for large  $E$  (i.e., neglecting terms  $\propto 1/E$ ):

$$\frac{1}{n} \frac{d^2 T}{d(\ln E)^2} + \left[ \frac{1}{\tau} + \xi(t, E) \right] \frac{dT}{d(\ln E)} + T \frac{d\xi(t, E)}{d(\ln E)} = 0. \quad (50)$$

<sup>6</sup> Notice the change of notation, we discuss formulas for energy  $E$  but results are for transverse momenta  $p_T$  here. However, they are taken at the midrapidity, i.e., for  $y \simeq 0$ , and for large transverse momenta,  $p_T > M$ , and in this region one has  $E = \sqrt{M^2 + p_T^2} \cosh(y) \simeq p_T$ .

Assuming now that noise  $\xi(t, E)$  increases logarithmically with energy,

$$\xi(t, E) = \xi_0(t) + \frac{\omega^2}{n} \ln E. \quad (51)$$

In this case Eq. (50) becomes an equation for the damped hadronic oscillator with solution in the form of log-periodic oscillation of temperature with frequency  $\omega$  and depending on initial conditions phase shift parameter  $\phi$ :

$$T = C \exp \left\{ -n \cdot \left[ \frac{1}{2\tau} + \frac{\xi(t, E)}{2} \right] \ln E \right\} \cdot \sin(\omega \ln E + \phi). \quad (52)$$

Averaging the noise fluctuations over time  $t$  and taking into account that the noise term cannot on average change the temperature,  $1/\tau + \langle \xi(t, E) \rangle = 0$ , one arrives at

$$T = \bar{a} + \frac{b'}{n} \sin(\omega \ln E + \phi). \quad (53)$$

This should now be compared with the parametrization of  $T(E)$  given by Eq. (46) and used to fit data in Fig.4<sup>7</sup>.

We close with the remark that, instead of using energy dependent noise  $\xi(t, E)$  given by Eq. (51) and keeping the relaxation time  $\tau$  constant. we could equivalently keep the energy independent white noise,  $\xi(t, E) = \xi_0(t)$ , but allow for the energy dependent relaxation time, for example in the form of

$$\tau = \tau(E) = \frac{n\tau_0}{n + \omega^2 \ln E}. \quad (54)$$

In this case the temperature evolution has the form

$$T(t) = \langle T \rangle + [T(t=0) - \langle T \rangle] E^{-t\omega^2/n} \exp \left( -\frac{t}{\tau_0} \right), \quad (55)$$

and  $T$  gradually approaches its equilibrium value  $\langle T \rangle$ . Actually, for  $\tau = \tau(E)$ , as in our case, this approach towards equilibrium is faster for large  $E$ . This is because, in addition to the usual exponential relaxation characteristic for  $\tau = \text{const}$  case, we have an additional factor  $\sim E^{-t\omega^2/n}$ .

## 6. Summary and conclusions

We presented examples of possible mechanisms resulting in quasi-power distributions exemplified by Tsallis distribution, Eq. (2). Our presentation had to be limited, therefore we did not touch thermodynamic connections

---

<sup>7</sup> Notice that only a small amount of  $T$ , of the order of  $\bar{b}/\bar{a} \sim 3\%$ , emerges from the stochastic process with energy dependent noise; the main contribution comes from the usual energy-independent Gaussian white noise.



of this distribution [8, 9] or the possible connection of Tsallis distributions with QCD calculations discussed recently [43, 44].

The main results presented here can be summarized in the following points:

- Statistical physics consideration, as well as "induced partition process", results in Eq. (15), i.e., in Tsallis distribution with  $q = (N - 3)/(N - 2) < 1$ . Fluctuations of the multiplicity  $N$  modify the parameter  $q$  which is now equal to  $q = 1 + Var(N)/\langle N \rangle^2 - 1/\langle N \rangle$ , cf. Eq. (30). Notice that conditional probability for the BG distribution again results in Eq. (15).
- Fluctuations of the multiplicity  $N$  are equivalent to results of an application of superstatistics, where the convolution

$$f(E) = \int g(T) \exp\left(-\frac{E}{T}\right) dT \quad (56)$$

becomes a Tsallis distribution, Eq. (2), for

$$g(T) = \frac{1}{\Gamma(n)T} \left(\frac{nT_0}{T}\right)^n \exp\left(-\frac{nT_0}{T}\right). \quad (57)$$

- Differentiating Eq. (56) one gets

$$\frac{df(E)}{dE} = -\frac{1}{T(E)}f(E) \quad \text{where} \quad T(E) = T_0 + \frac{E}{n}. \quad (58)$$

This is nothing else than a "preferential attachment" case, again resulting in a Tsallis distribution which for  $T(E) = T_0$  becomes a BG distribution, cf. Eq. (5).

- Replacing in Eq. (58) differentials by finite differences, cf. Eq. (43), one gets for  $\delta E = \alpha n T(E)$  the scale invariant relation, Eq. (44), which results in log-periodic oscillations in Tsallis distributions <sup>8</sup>.

In addition to this line of reasoning, we have also brought in the problem of the apparent duality between the nonextensive parameters obtained from the whole phase space measurements of multiplicity and more local

---

<sup>8</sup> Among numerous other explanations we can therefore say that we have demonstrated that a Tsallis distribution, which can be regarded as generalization to real power  $n$  of such well known distributions as the Snedecor distribution (with  $n = (\nu + 2)/2$  with integer  $\nu$ , for  $\nu \rightarrow \infty$  it becomes an exponential distribution), can be extended to complex nonextensivity parameter.

measurements of transverse momenta. This point deserves an experimental and phenomenological scrutiny.

Finally, we tentatively suggested that, by choosing the right constraints, which account for additive or multiplicative processes considered, one can also get a Tsallis distribution directly from the Shannon information entropy.

### Acknowledgment

This research was supported in part by the National Science Center (NCN) under contract DEC-2013/09/B/ST2/02897. We would like to warmly thank Dr Eryk Infeld for reading this manuscript.

### REFERENCES

- [1] V. Khachatryan et al. (CMS Collaboration), JHEP**02**, 041 (2010); V. Khachatryan et al. (CMS Collaboration), Phys. Rev. Lett. **105** (2010) 022002.
- [2] K. Aamodt *et al.* (ALICE Collaboration), Phys. Lett. B **693** (2010) 53; Eur. Phys. J. C **71** (2011) 1594 and 1655.
- [3] G. Aad *et al.* (ATLAS Collaboration), New J. Phys. **13** (2011) 053033.
- [4] C. Michael and L. Vanryckeghem, J. Phys. G **3** (1977) L151; C. Michael, Prog. Part. Nucl. Phys. **2** (1979) 1.
- [5] G. Arnison et al (UA1 Collab.), Phys. Lett. B **118** (1982) 167.
- [6] R. Hagedorn, Riv. Nuovo Cimento **6** (1984) 1.
- [7] C. Tsallis, J. Stat. Phys. **52** (1988) 479 and Eur. Phys. J. A **40** (2009) 257; cf. also C. Tsallis, *Introduction to Nonextensive Statistical Mechanics* (Berlin 2009: Springer). For an updated bibliography on this subject, see <http://tsallis.cat.cbpf.br/biblio.htm>.
- [8] J. Cleymans, D. Worku, J. Phys. G **39** (2012) 025006 and Eur. Phys. J. A **48** (2012) 160; A. Deppman, J. Phys. G **41** (2014) 055108.
- [9] O. J. E. Maroney, Phys. Rev E **80** (2009) 061141; T. S. Biró, K. Ürmösy and Z. Schram, J. Phys. G **37** (2010) 094027; T. S. Biró and P. Ván, Phys. Rev. E **83** (2011) 061147; T. S. Biró and Z. Schram, Eur. Phys. J. Web Conf. **13**, (2011) 05004; T. S. Biró, *Is there a Temperature? Conceptual Challenges at High Energy, Acceleration and Complexity* (Springer, New York Dordrecht Heidelberg London, 2011); P. Ván, G. G. Barnaföldi, T. S. Biró and K. Ürmösy, J. Phys.: Conf. Ser. **394** (2012) 012002.
- [10] C. Beck, E. G. D. Cohen, Physica A **322**(2003) 267; F. Sattin, Eur. Phys. J. B **49** (2006) 219.
- [11] G. Wilk, Z. Włodarczyk, Phys. Rev. Lett. **84** (2000) 2770 and Chaos, Solitons, Fractals **13** (2002) 581.
- [12] T. S. Biró, A. Jakovác, Phys. Rev. Lett. **94** (2005) 132302.
- [13] G. Wilk, Z. Włodarczyk, Eur. Phys. J. A **40** (2009) 299.

- [14] G. Wilk, Z. Włodarczyk, Phys. Rev. C **79** (2009) 054903.
- [15] G. Wilk, Z. Włodarczyk, Cent. Eur. J. Phys. **8** (2010) 726.
- [16] G. Wilk, Z. Włodarczyk, Acta Phys. Pol. B **35** (2004) 871 and 2141; C. Tsallis, Eur. Phys. J. ST **161** (2008) 175; D. J. B. Soares, C. Tsallis, A. M. Mariz, L. R. da Silva, Europhys. Lett. **70** (2008) 70.
- [17] C. Anteneodo, C. Tsallis, J. Math. Phys. **44** (2003) 5194.
- [18] G. Wilk, Z. Włodarczyk, AIP Conf. Proc. **1558** (2013) 893.
- [19] G. Wilk, Z. Włodarczyk, Physica A **390** (2011) 3566.
- [20] W. Feller, *An introduction to probability theory and its applications*, Vol. II, John Wiley and Sons Inc., New York 1966.
- [21] M. Sibata, Phys. Rev. D **24** (1981) 1847; C. Y. Wong, Phys. Rev. D **30** (1988) 972.
- [22] G. Wilk, Z. Włodarczyk, Eur. Phys. J. A **48** (2012) 161.
- [23] F. Reif, *Statistical Physics*, (McGraw-Hill Book Company, New York (1967)); R. P. Feynman, *Statistical Mechanics. A set of Lectures*, ( W.A. Benjamin Inc. Menlo Park, California, 1972).
- [24] G. Wilk, Z. Włodarczyk, Phys. Lett. B **727** (2013) 163.
- [25] P. Carruthers, C. S. Shih, Int. J. Mod. Phys. A **2** (1986) 1447; C. Vignat, A. Plastino, Phys. Lett. A **360** (2007) 415; R. A. Fisher, Ann. Eugenics **11** (1941) 182.
- [26] T. S. Biró, G. G. Barnaföldi, P. Van, Physica A **417** (2015) 215; T. S. Biró, P. Van, G. G. Barnaföldi, K. Ürmösy, *Statistical Power Law due to Reservoir Fluctuations and the Universal Thermostat Independence Principle*, arXiv:1409.5975[cond-mat.stat-mech], to be published in Entropy.
- [27] G. Aad et al. (ATLAS Collaboration), Phys. Rev. D **84** (2011) 054001.
- [28] E. Rufeil Fiori, A. Plastino, Physica A **392** (2013) (2013).
- [29] S. M. Papalexiou, D. Koutsoyiannis, *Entropy maximization, p-moments and power-type distributions in nature*, available at <http://itia.ntua.gr/1127>.
- [30] A. Rostovtsev, *On a geometric mean and power-law statistical distributions*, arXiv:cond-mat/0507414 (2005).
- [31] G. Wilk, Z. Włodarczyk, Cent. Eur. J. Phys. **10** (2012) 568.
- [32] A. Białas, M. Bleszyński, W. Czyż, Nucl. Phys. B **111** (1976) 461.
- [33] C. Geich-Gimbel, Int. J. Mod. Phys. A **4** (1989) 1527; A. K. Dash, B. M. Mohanty, J. Phys. G **37** (2010) 025102.
- [34] T. Wibig, J. Phys. G **37** (2010) 115009.
- [35] C. Alt et al., Phys. Rev. C **77** (2008) 024903 and 034906 (2008); S. V. Afanasiew et al., Phys. Rev. C **66** (2002) 054902.
- [36] B. B. Back et al. (PHOBOS Collaboration), Phys. Rev. C **74** (2006) 021902.
- [37] Y. Huang, H. Saleur, C. Sammis, D. Sornette, Europhys. Lett. **41** (1998) 43; H. Saleur, C.G. Sammis, D. Sornette, J. Geophys. Res. **101** (1996) 17661; A. Krawiecki, K. Kacperski, S. Matyjaskiewicz, J. A. Holyst, Chaos Solitons Fractals **18** (2003) 89; J. Bernasconi, W. R. Schneider, J. Stat. Phys. **30** (1983)

- 355; D. Stauffer, D. Sornette, *Physica A* **252** (1998) 271; D. Stauffer, *Physica A* **266** (1999) 35.
- [38] S. Chatrchyan et al. (CMS Collaboration), *Eur. Phys. J. C* **72** (2012) 1945.
  - [39] B. Abelev et al. (ALICE Collaboration), *Phys. Lett. B* **720** (2012) 52.
  - [40] G. Wilk, Z. Włodarczyk, *Physica A* **413** (2014) 53.
  - [41] G. Wilk, Z. Włodarczyk, *Log-periodic oscillations of transverse momentum distributions*, arXiv:1403.3508v2.
  - [42] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry* (Elsevier Science Publishers B.V., North-Holland, Amsterdam, 1987), Chap. VIII.
  - [43] C. Y. Wong, G. Wilk, *Acta Phys. Pol. B* **43** (2012) 2047; *Phys. Rev. D* **87** (2013) 114007 and *Relativistic Hard-Scattering and Tsallis Fits to  $p_T$  Spectra in  $pp$  Collisions at the LHC*, arXiv:1309.7330[hep-ph], to be published in *The Open Nuclear and Particle Physics Journal* (2014).
  - [44] L. J. L. Cirto, C. Tsallis, C.-Y. Wong, G. Wilk, *The transverse-momenta distributions in high-energy  $pp$  collisions - A statistical-mechanical approach*, arXiv:1409.3278[hep-ph] and C.-Y. Wong, G. Wilk, L.J. L. Cirto, C. Tsallis, *Possible Implication of a Single Nonextensive  $p_T$  Distribution for Hadron Production in High-Energy  $pp$  Collisions*; arXiv:1412.0474[hep-ph].